Boundary Value Problems for Second-Order Differential Operator Equations*

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ABSTRACT

It is proved that the resolution problem of an operator boundary-value problem for a second-order differential operator equation with constant coefficients is solved in terms of solutions of certain algebraic operator equations. Explicit expressions of solutions are given.

1. INTRODUCTION

Through this paper, H will denote a separable, complex Hilbert space, and $L(H)$ will denote the algebra of all bounded linear operators on H. If T is in $L(H)$, $\sigma(T)$ denotes the spectrum of T, and $\sigma_\pi(T)$ denotes its approximate point spectrum. Also $\sigma_\delta(T)$ denotes its approximate defect spectrum, defined by $\sigma_\delta(T)=\{\lambda\in\mathbb{C};\ T-\lambda \text{ is not onto}\}$. If T^* denotes the adjoint operator of T, then [10, §4.7]

$$\sigma_\pi(T)=\sigma_\delta(T^*),\qquad \sigma_\delta(T)=\sigma_\pi(T^*).$$

It is well known that the resolution problem of a scalar second-order linear differential equation with constant coefficients,

$$\ddot{x}(t)+a_1\dot{x}(t)+a_0x(t)=0,$$

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is solved by the determination of the roots of the associate algebraic equation

$$\lambda^2 + a_1\lambda + a_0 = 0.$$

The resolution problem of the operator differential equation

$$\ddot{X}(t) + A_1\dot{X}(t) + A_0X(t) = 0 \tag{1.1}$$

appears in the analysis of vibrational systems [3]. In this paper we consider the resolution problem of boundary-value problems for Equation (1.1). In analogous way to the scalar case, we obtain sufficient conditions for these resolution problems, from the existence of a solution of the algebraic operator equation

$$X^2 + A_1X + A_0 = 0. \tag{1.2}$$

The resolution problem (1.2) is related to the linear factorization of the polynomial operator

$$L(\lambda) = \lambda^2 + \lambda A_1 + A_0.$$

For the finite-dimensional case, it is known [3] that we can obtain a factorization $L(\lambda) = (\lambda I + X_1)(\lambda I + X_2)$ when the companion operator

$$C_L = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix}$$

of $L(\lambda)$ is diagonable. The infinite-dimensional case is treated in a more general context in [7].

It is clear that if $L(\lambda) = (\lambda I + X_1)(\lambda K + X_2)$, then $-X_2$ is a solution of (1.2). Otherwise, Equation (2.1) can be unsolvable. In fact, if $A_1 = 0$ and the operator $-A_0$ has no square root, (for instance, if $-A_0$ is an injective unilateral weighted shift [8, p. 63]), then (1.2) is unsolvable.

2. ON THE OPERATOR EQUATION $X^2 + A_1X + A_0 = 0$

This section is concerned with the resolution problem of Equation (2.1). By application of annihilating functions of the coefficient operators, this equation is reduced to a linear operator equation of the type $B_1X + B_0 = 0$.

Moreover, in the finite-dimensional case, every algebraic operator [6, p. 63] is annihilated by a polynomial. It is easy to show that an injective unilateral weighted shift on an infinite-dimensional Hilbert space H is not annihilated by any nontrivial polynomial.

Let D denote the open unit disc in the complex plane, and let H^∞ denote the Banach algebra of all bounded, analytic functions on D, under the supremum norm. If T is any completely nonunitary contraction on H, one has the Sz.-Nagy–Foias functional calculus Φ_T defined on H^∞, $\Phi_T: H^\infty \to L(H)$, and for the class C_0 of those completely nonunitary contractions T, there exist a nonzero function $f \in H^\infty$ such that $f(T) = 0$ [5, p. 123]. In [1] and [9] are given several classes of operators which are annihilated by analytic functions.

THEOREM 1. *Let X be a solution of the operator equation* (1.2), *and let* $\alpha = \|A_0\| + \|A_1\|$. *If* $f(z) = \sum_{n=0} a_n z^n$ *is an analytic function in the disc* $|z| < \rho$, *where* $1 < \rho$, $\alpha < \rho$, *and* $f(x) = 0$, *then X satisfies the linear operator equation*

$$B_1 X + B_0 = 0, \tag{2.1}$$

where

$$\begin{aligned} B_j &= \sum_{n \geq 0} a_n Q_{n,j}, \qquad j = 0,1, \\ [Q_{0,0}; Q_{0,1}] &= [0; I], \\ Q_{n,j} &= Q_{n-1,j-1} - Q_{n-1,1} A_j, \\ Q_{n-1,-1} &= 0 \qquad \textit{for all} \quad n \geq 1. \end{aligned} \tag{2.2}$$

Proof. Let $Q_n(X)$ be the operators defined by

$$\begin{aligned} X &= Q_0(X), \\ X^2 &= -A_0 - A_1 X = Q_1(X), \\ X^3 &= X^2 X = -A_0 X - A_1 X^2 = Q_2(X), \\ &\vdots \\ X^{2+n-1} &= Q_n(X). \end{aligned} \tag{2.3}$$

It is easy to show that

$$Q_n(X) = Q_{n,0} + Q_{n,1}X, \qquad n \geq 0, \tag{2.4}$$

where $Q_{n,j}$ can be obtained recurrently from the coefficient operators A_j and from (2.2). In fact, from (2.1), (2.3), and (2.4) it follows that

$$\begin{aligned} Q_n(X) &= X^{2+n-1} = X^{2+n-2}X = Q_{n-1}(X)X = (Q_{n-1,0} + Q_{n-1,1}X)X \\ &= Q_{n-1,0}X + Q_{n-1}X^2 = -Q_{n-1,1}A_0 + (Q_{n-1,0} - Q_{n-1,1}A_1)X. \end{aligned}$$

Considering the expression of $Q_n(X)$, given by (4.2), we have

$$Q_{n,0} = -Q_{n-1,1}A_0, \qquad Q_{n,1} = Q_{n-1,0} - Q_{n-1,1}A_1.$$

Thus $[Q_{n,0}; Q_{n,1}] = [Q_{n-1,0}; Q_{n-1,1}]C_L$ for all $n \geq 1$, and from (2.2) we obtain

$$[Q_{n,0}; Q_{n,1}] = [0; I]C_L^n, \qquad n \geq 1. \tag{2.5}$$

By the hypothesis, the operator series $\sum_{n \geq 0} a_n (C_L)^n$ is norm convergent in $L(X \oplus X)$. From (2.5) the convergence of the operator series

$$\sum_{n \geq 0} a_n Q_{n,j}, \qquad j = 0, 1,$$

follows. Postmultiplying the expression $f(X) = \sum_{n \geq 0} a_n X^n = 0$ by X and considering (2.3) and (2.4), we obtain

$$\begin{aligned} 0 &= \sum_{n \geq 0} a_n X^{n+1} = \sum_{n \geq 0} a_n Q_n(X) = \sum_{n \geq 0} a_n (Q_{n,0} + Q_{n,1}X) \\ &= \sum_{n = 0} a_n Q_{n,0} + \left(\sum_{n \geq 0} a_n Q_{n,1} \right) X = B_1 X + B_0. \end{aligned}$$ ■

the following result is a converse of Theorem 1.

THEOREM 2. *Let $f(z) = \sum_{n \geq 0} a_n z^n$ be an analytic function which satisfies the properties of Theorem* 1, *such that*

$$f(C_L) = f\left(\begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix} \right) = \begin{bmatrix} B_{0,1} & B_{1,1} \\ B_0 & B_1 \end{bmatrix}$$

and B_1 *is invertible with* $B_{0,1} = B_{11}B_1^{-1}B_0$. *Then any solution of* (2.1) *satisfies* (1.2)

Proof. From the hypothesis, the operator $f(C_L)$ is well defined, and we are going to see that any solution X of Equation (2.1) is also a solution of the equation

$$B_{0,1} + B_{1,1}X = 0.$$

From the hypothesis $B_{0,1} = B_{1,1}B_1^{-1}B_0$, premultiplying the equation (1.2) by $B_{1,1}B_1^{-1}$, and substituting, it follows that X satisfies the equation

$$B_{0,1} + B_{1,1}X = 0. \tag{2.6}$$

From the commutativity between $f(C_L)C_L$ and $C_L f(C_L)$, and equating the last rows of these operators, it follows that

$$-B_1B_0 = -A_0B_{0,1} - A_1B_0,$$

$$B_0 - B_1A_1 = -A_0B_{1,1} - A_1B_1,$$

from which we obtain

$$A_1B_j = B_1A_j - A_0B_{j,1} - B_{j-1}, \qquad j = 0,1, \tag{2.7}$$

with the convention that $B_{-1} = 0$.

Premultiplying equation (1.2) by A_1 and from (2.7), it follows that

$$A_1B_0 + A_1B_1X = 0,$$

$$\sum_{j=0}^{1} (B_1A_j - A_0B_{1,1} - B_{j-1})X^j = 0, \tag{2.8}$$

$$B_1A_0 - A_0B_{0,1} + (B_{1,1} - A_0B_{1,1} - B_0)X = 0.$$

Considering (2.6), it follows that $B_{0,1} = -B_{1,1}X$, and sustituting this expression into (2.8) yields

$$B_1A_0 + B_1A_1X - B_0X = 0. \tag{2.9}$$

Postmultiplying Equation (1.2) by X, it follows that $B_1X^2 = -B_0X$, and after substitution into (2.9), it is deduced that

$$B_1(A_0 + A_1X + X^2) = 0.$$

Premultiplying by B_1^{-1}, the result is concluded. ■

3. BOUNDARY-VALUE PROBLEMS FOR THE EQUATION $\ddot{X}(T) + A_1\dot{X}(T) + A_0X(T) = 0$

We begin this section considering the Cauchy problem

$$\ddot{X}(t) + A_1\dot{X}(t) + A_0X(t) = 0,$$
$$X(0) = C_0, \qquad \dot{X}(0) = C_1, \tag{3.1}$$

where C_i and A_i, $i = 0, 1$, are bounded linear operators on a separable Hilbert space H. Making $X = Y_1$, $\dot{X} = Y_2$, the resolution problem is reduced to a linear differential operator system on $H \oplus H$:

$$\dot{Y}(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} \qquad Y(0) = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}. \tag{3.2}$$

LEMMA 1. *If X_0 is a solution of Equation* (2.1), *then a solution of the problem* (3.1) *is given by the expression*

$$X(t) = \exp(X_0t)C_0 + \int_0^t \exp[X_0(t-u)]\exp(uX_1)D\,du, \tag{3.3}$$

where $X_1 = -(X_0 + A_1)$ and $D = C_1 - X_0C_0$.

Proof. The system (3.2) can be expressed as

$$\dot{Y}(t) = C_LY(t), \qquad Y(0) = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}.$$

Let J be the operator

$$J = \begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix}.$$

It is straightforward to prove

$$J^{-1} C_L J = W = \begin{bmatrix} X_0 & I \\ 0 & X_1 \end{bmatrix},$$

and considering the transformation $Y(t) = JZ(t)$, it follows that

$$Y(0) = JZ(0), \qquad Z(0), \qquad Z(0) = J^{-1} Y(0) = \begin{bmatrix} C_0 \\ C_1 - X_0 C_0 \end{bmatrix},$$

and thus the initial problem is reduced to the following linear system:

$$Z(t) = \begin{bmatrix} Z_1(t) \\ Z_2(t) \end{bmatrix}, \quad \dot{Z}(t) = \begin{bmatrix} X_0 & I \\ 0 & X_1 \end{bmatrix} Z(t), \qquad Z(0) = \begin{bmatrix} C_0 \\ C_1 - X_0 C_0 \end{bmatrix}.$$

By decomposition of last system we obtain

$$\begin{aligned} \dot{Z}_1(t) &= X_0 Z_1(t) + Z_2(t), \qquad & Z_1(0) &= C_0, \\ \dot{Z}_2(t) &= X_1 Z_2(t), & Z_2(0) &= C_1 - X_0 C_0. \end{aligned} \tag{3.4}$$

Solving (3.4) we obtain $Z_2(t) = \exp(tX_1) D$, and substituting in the first equation of (3.4) and solving, we have

$$Z_1(t) = \exp(X_0 t) C_0 + \int_0^t \exp[X_0(t-u)] \exp(X_1 u) D \, du. \tag{3.5}$$

From (3.5), (3.4), and $Y(t) = JZ(t)$, it follows that

$$X(t) = Y_1(t) = Z_1(t), \qquad X(0) = Z_1(0) = C_0,$$

$$\dot{X}(0) = X_0 Z_1(0) + Z_2(0) = X_0 C_0 + C_1 - X_0 C_0 = C_1,$$

and the result is proved. ■

The following theorem solves a boundary-value problem related to (1.1).

THEOREM 3. *We consider the problem*

$$\begin{aligned} \ddot{X}(t) + A_1\dot{X}(t) + A_0X(t) &= 0, \\ F\dot{X}(\alpha) - X(0)G &= E, \end{aligned} \tag{3.6}$$

where the coefficients are bounded linear operators on H, and $\alpha > 0$. If X_0 is a solution of Equation (2.1), *and*

$$\sigma_\delta(FX_0 \exp(\alpha X_0)) \cap \sigma_\pi(G) = \varnothing, \tag{3.7}$$

then the problem (3.6) *is solvable and a solution is given by*

$$X(t) = \exp(X_0 t)\, C_0, \qquad t \in [0, \alpha],$$

where C_0 is a solution of algebraic equation

$$AX - XB = C$$

and

$$A = FX_0 \exp(\alpha X_0), \quad B = G, \quad C = E - F\exp(\alpha X_1), \quad X_1 = -(X_0 + A_1).$$

Proof. Given X_0, from Lemma 1 it follows that a solution of the problem (3.1) has the expression (3.3). Taking $C_1 = X_0C_0$, it results that $D = 0$ and

$$X(t) = \exp(tX_0)\, C_0, \qquad t \in [0, \alpha], \tag{3.8}$$

where $C_0 = X(0)$. From (3.8) and by differentiation, it follows that

$$\dot{X}(t) = \exp(tX_0)\, X_0C_0 = X_0 \exp(tX_0)\, C_0 = X_0X(t). \tag{3.9}$$

From here, the boundary-value problem of (3.6) is verified if there exist a solution C_0 for the equation

$$FX_0 \exp(\alpha X_0)C_0 - C_0G = E. \tag{3.10}$$

From Theorem 5 of [2, pp. 1387] and the hypothesis (3.7), the equation (3.10) is solvable. We conclude that $X(t)$ given by (3.8), where X_0 is a solution of (2.1) and C_0 is a solution of (3.10), is a solution of the boundary problem (3.6). ∎

In order to obtain an explicit solution of (3.6), it is clear from (3.8) that it is necessary to find an explicit expression for a solution of Equation (3.10). For the finite-dimensional case, such an expression is available.

COROLLARY 1. *If H is finite-dimensional, and one replaces the hypothesis* (3.7) *in Theorem* 3 *with the hypothesis $\sigma(FX_0\exp(\alpha X_0))\cap\sigma(G)=\varnothing$, then a solution of* (3.6) *is given by* (3.8), *where C_0 has the expression*

$$C_0=\left(\sum_{k=0}^{n}p_k[FX_0\exp(\alpha X_0)]^k\right)^{-1}$$
$$\times\left(\sum_{k=1}^{n}\sum_{j=1}^{k}p_j[FX_0\exp(\alpha X_0)]^{j-1}EG^{k-j}\right),$$

where $p(\lambda)=\sum_{k=0}^{n}p_k\lambda^k$, the characteristic polynomial of G.

Proof. The result is a consequence of Theorem 3 and [4]. ■

REMARK 3.1. Note that for the finite-dimensional case, a solution X_0 of Equation (2.1) is available if the companion operator C_L is diagonable [1], and thus from this hypothesis and (3.7), an explicit solution of the problem (3.6) is given by Corollary 1.

In the following we consider a different boundary.

THEOREM 4. *If X_0 is a solution of Equation* (2.1) *and F,G,E, and $\alpha>0$ satisfy the property*

$$\sigma_\delta(F\exp(\alpha X_0))\cap\sigma_\pi(G)=\varnothing, \tag{3.11}$$

then the problem

$$\begin{gathered}\ddot{X}(t)+A_1\dot{X}(t)+A_0X(t)=0,\\ FX(\alpha)-X(0)G=E\end{gathered} \tag{3.12}$$

is solvable, and a solution is given by

$$X(t)=\exp(tX_0)C_0,\qquad t\in[0,\alpha], \tag{3.13}$$

where C_0 is a solution of

$$(F\exp(\alpha X_0))X-XG=E. \tag{3.14}$$

Proof. Taking $C_1 = X_0C_0$ in Lemma 1, the expression (3.13) yields a solution of Equation (1.1). $X(t)$ satisfies the boundary condition given by (3.12) if there exists a solution of Equation (3.14), because $FX(\alpha) - X(0)G = F\exp(\alpha X_0)C_0 - C_0G$. Moreover from Theorem 5 of [2, p. 1387], and from the hypothesis (3.11), Equation (3.14) is solvable. ∎

COROLLARY 2. *If H is finite-dimensional, X_0 is a solution of* (1.2), *and F, G, X_0, and $\alpha > 0$ satisfy the hypothesis $\sigma(F\exp(X_0\alpha)\cap\sigma(G) = \varnothing$, then $X(t)$ given by* (3.13) *satisfies* (3.12), *where C_0 is*

$$C_0 = \left(\sum_{k=0}^{n} p_k[F\exp(\alpha X_0)]^k\right)^{-1}\left(\sum_{k=1}^{n}\sum_{j=1}^{k} p_j[F\exp(\alpha X_0)]^{j-1}EG^{k-j}\right),$$

and $p(\lambda) = \sum_{k=0}^{n} p_k\lambda^k$ is the characteristic polynomial of G.

Proof. The result is an easy consequence of Theorem 4 and [4]. ∎

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